

Amenability notions of hypergroups and some applications to locally compact groups

MAHMOOD ALAGHMANDAN

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Abstract

Different notions of amenability of hypergroups and the relations between them are studied. Developing Leptin's theorem for hypergroups, we characterize the existence of a bounded approximate identity for the Fourier algebra of hypergroups. We study the Leptin condition for some classes of hypergroups derived from representation theory of compact groups. Studying amenability of the hypergroup algebra of discrete commutative hypergroups, we apply these hypergroup tools to derive some results on amenability properties of some Banach algebras on locally compact groups. We prove some results concerning amenability of $ZA(G) = A(G) \cap ZL^1(G)$ for compact groups G and amenability of $Z\ell^1(G)$ for FC groups G . Also we show that proper Segal algebras of compact connected simply connected real Lie groups are not approximately amenable.

Keyword: hypergroups; Fourier algebra; amenability; compact groups.

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Amenability notion of locally compact groups has different characterizations which lead to different definitions. For example, amenability of locally compact groups is equivalent to a family of structural properties called [Følner type conditions](#). Also subsequently, the amenability leads to the existence of a bounded approximate identity of the Fourier algebra of the group. And as Johnson, [25], proved they are equivalent to the existence of a virtual diagonal for the group algebra. Leptin, [31], showed that the Følner type conditions on a group admit a bounded approximate identity of the Fourier algebra. Indeed, the existence of a bounded approximate identity is equivalent to the amenability of the underlying group. This result is known as [Leptin's theorem](#). So, although we experience a variety in the definitions, eventually a unity in the notion emerges (unlike hypergroups as we observe in the following).

Similar to the amenability of groups, different notions of amenability have been defined for hypergroups. Skantharajah, [41], defined the actual concept of [amenability](#) in the sense of the existence of a left invariant mean. According to this definition of amenability, large classes of hypergroups are amenable including all commutative hypergroups and compact hypergroups.

But unlike groups, the existence of a left invariant mean does not imply the amenability of the corresponding hypergroup algebra, [41], while the other side of the Johnson's theorem is still true for hypergroups that is, the amenability of the hypergroup algebra implies the existence of a left invariant mean. Skantharajah also defined (approximately invariant) [Reiter's nets](#) in the hypergroup algebra, denoted by (P_1) , and the Banach space of all square integrable functions on a hypergroup, denoted by (P_2) . He showed that (P_2) implies (P_1) and (P_1) is equivalent to the amenability. These notions were also defined and studied for [fusion algebras](#) which form discrete hypergroups, [19].

The [Leptin condition](#) (as a specific case of the Følner conditions) for hypergroups first was defined by Singh in [40] to study the norm of positive integrable functions as operators over the L^2 -space of a hypergroup. After a long gap, the author, in [1], independently defined the Leptin condition on hypergroups and studied its application to the hypergroup structure defined on duals of compact groups.

In this paper, we investigate different amenability notions of hypergroups and their relations. First, we look closer at a generalization of Følner type conditions over hypergroups. This generalization not only investigates some properties of particular classes of hypergroups, but it also improves our knowledge about the Fourier algebra of hypergroups. We should note that the Fourier space is in general just a Banach space. But for wide classes of hypergroups –specially the hypergroup structures which are admitted by groups– this Banach space actually forms a Banach algebra, [34, 35]. This class of hypergroups are called [regular Fourier hypergroups](#). We study the existence of a bounded approximate identity of the Fourier algebra for the class of regular Fourier hypergroups. The outcome is Leptin's theorem for hypergroups which shows that a bounded approximate identity can always be normalized to 1 and its existence is equivalent to (P_2) . We also study amenability of hypergroup algebras for discrete commutative hypergroups satisfying (P_2) . We show that for this class of hypergroups, the hypergroup algebra cannot be amenable (as a Banach algebra) if the Haar measure goes to infinity (as a weight on the hypergroup).

Some of the main examples of hypergroup structures are mathematical objects derived from locally compact groups. These hypergroup instances highlight a hypergroup perspective towards some Banach algebras on locally compact groups. For example, the center of group algebras for locally compact groups with relatively compact conjugacy classes are hypergroup algebras. Also, the subalgebra of the Fourier algebra on compact groups which consists of all class functions, denoted by $ZA(G)$, is a hypergroup algebra. These close ties not only emphasize the applications that this hypergroup study has for locally compact groups but also give us a rich class of examples on which we observe our theory. In this paper, we apply hypergroup machinery to study some amenability properties of these algebras.

This paper is organized as follows. In Section 1, we introduce the notations and present some facts regarding hypergroups and their Fourier spaces that will be used later. One may note that the Fourier space of hypergroups, even as a Banach space, carries interesting properties, [27, 43]. Here we show that similar to locally compact groups, every element of the Fourier space is the convolution of two square integrable functions. Section 2 introduces and studies Følner

type conditions, in particular the Leptin condition for hypergroups. First in Subsection 2.1, we define different Følner type conditions on hypergroups and consider their co-relations. In an attempt to answer a question about Segal algebras on compact groups, the author, in [1], defined the Leptin condition for hypergroups. Here, we expand that study by defining more conditions. These conditions on one hand help us to study the behaviour of hypergroup Haar measures analogous to groups. On the other hand, they develop a criterion to vaguely measure the growth rate of the hypergroup action between subsets of a hypergroup. The question of approximate amenability of Segal algebras of compact groups highlights the importance of the Leptin condition for the class of discrete hypergroups defined on dual of compact groups, [1]. Hence, in Subsection 2.2, we apply some studies on the irreducible decomposition of tensor products on representations of compact groups to consider the Leptin condition for the dual hypergroup structure of some classes of compact groups including $SU(n)$ for $n \geq 2$.

The equivalence of the Leptin condition and the existence of a bounded approximate identity of Fourier algebras is a crucial part of the theory of amenable groups. In Section 3, we apply the notion of the hypergroup Leptin condition to study the existence of a bounded approximate identity for the Fourier algebra of regular Fourier hypergroups. (Following [34], we call a hypergroup **regular Fourier** when its Fourier space forms a Banach algebra.) In Subsection 3.1, we characterize the existence of a bounded approximate identity in the Fourier algebra of these hypergroups with respect to the amenability notion (P_2) . This characterization is an analog of **Leptin's theorem**. Although (P_2) implies the amenability of a hypergroup, it is strictly stronger. We apply the hypergroup Leptin's theorem to double coset hypergroups and conclude (P_2) for them when they are admitted by amenable locally compact groups. We close the section by a brief study on the existence of bounded approximate identities in ideals of Fourier algebras for regular Fourier hypergroups in Subsection 3.2.

Johnson, in [25], proved that the amenability of groups is equivalent to the amenability of their group algebras. For hypergroups, it is known that the amenability of a hypergroup algebra implies the existence of a left invariant mean but not necessarily vice versa, [41]. The question of amenability of hypergroup algebras is a challenging question and so far most of the partial answers are concerning specific classes of hypergroups, especially polynomial hypergroups, (see [30, 29]). In Section 4, we study the amenability of the hypergroup algebra of discrete commutative hypergroups with property (P_2) . As we mentioned before, this result and the appearance of (P_2) in hypergroup Leptin's condition emphasize the importance of this amenability notion of (P_2) . Interestingly, (P_2) is even equivalent to a **Kesten** type condition for fusion algebras, see [19] where (P_2) is even called the **amenability** of a fusion algebra while the actual concept of amenability is called the **weak amenability**. We close this section by studying the amenability of a class of multivariable polynomial hypergroups known as **multivariable Chebychev polynomial hypergroups**. Here we apply a significantly shorter proof to extend a result of [30] which was for one variable version of these hypergroups.

Due to the fact that locally compact groups have strong ties to hypergroups, our results in the preceding sections have applications to groups; Section 5 presents some of these applications. First in Subsection 5.1, applying the information obtained from the Leptin condition on duals

of compact groups, we extend the main result of [1]; every proper Segal algebra on compact connected simply connected real Lie groups is not approximately amenable. Subsection 5.2 studies the amenability of $ZA(G)(= A(G) \cap ZL^1(G))$ for a compact group G with respect to some conditions on the dimensions of the irreducible unitary representations of G . The main result of this subsection is an analog of a result by Johnson in [25] which shows that $ZA(G)$ is not amenable for **tall groups**. Eventually, Subsection 5.3, studies the amenability of the center of the group algebra for (infinite discrete) **FC groups**. A discrete group G is called FC or **finite conjugacy** if every conjugacy class of G is finite. Here we observe that if for every integer n there are just finitely many conjugacy classes with the cardinality n , then the center of the group algebra is not amenable. This subsection follows a previous paper of the author with Yemon Choi and Ebrahim Samei, [3], which concerned the amenability of a specific class of FC groups.

Some results of this paper are based on work from the author's Ph.D. thesis, [2], under the supervision of Yemon Choi and Ebrahim Samei.

We warn the reader that many facts which are either immediate or well known for Følner conditions and other amenability notions of (amenable) groups and their Fourier algebras are unknown for regular Fourier hypergroups. One may look at [2, 41] for some counterexamples. Here we also highlight some critical differences that one should consider when one generalizes group ideas to hypergroups in some remarks in the manuscript.

1 Preliminaries and notation

Since some results in this paper may target people who are not deeply engaged in the theory of hypergroups, we present a rather detailed background here.

1.1 Hypergroups

For notations, definitions, and properties of hypergroups, we mainly cite [7]. As a short summary for hypergroups, we may present the following.

Definition 1.1 [7, 1.1.2]

We call a locally compact space H a **hypergroup** if the following conditions hold.

- (H1) There exists an associative binary operation $*$ called the **convolution** on $M(H)$ under which $M(H)$ is an algebra. Moreover, for every x, y in H , $\delta_x * \delta_y$ is a positive measure with compact support and $\|\delta_x * \delta_y\|_{M(H)} = 1$.
- (H2) The mapping $(x, y) \mapsto \delta_x * \delta_y$ is a continuous map from $H \times H$ into $M(H)$ equipped with the weak* topology that is $\sigma(M(H), C_c(H))$.
- (H3) The mapping $(x, y) \rightarrow \text{supp}(\delta_x * \delta_y)$ is a continuous mapping from $H \times H$ into $\mathfrak{L}(H)$ equipped with the Michael topology (see [7, 1.1.1]).

- (H4) There exists an element (necessarily unique) e in H such that for all x in H , $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$.
- (H5) There exists a (necessarily unique) homeomorphism $x \rightarrow \tilde{x}$ of H called **involution** satisfying the following:
- (i) $(\tilde{\tilde{x}}) = x$ for all $x \in H$.
 - (ii) If \check{f} is defined by $\check{f}(t) := f(\tilde{t})$ for all $f \in C_c(H)$ and $t \in H$, one may define $\check{\mu}(f) := \mu(\check{f})$ for all $\mu \in M(H)$. Then $(\delta_x * \delta_y)^\check{} = \delta_{\check{y}} * \delta_{\check{x}}$ for all $x, y \in H$.
- (H6) e belongs to $\text{supp}(\delta_x * \delta_y)$ if and only if $y = \tilde{x}$.

Let $(H, *, \sim)$ be a (locally compact) **hypergroup**. The notation $A * B$ stands for $\cup \{\text{supp}(\delta_x * \delta_y) : \text{for all } x \in A, y \in B\}$ for A, B subsets of the hypergroup H . By abuse of notation, we use $x * A$ to denote $\{x\} * A$. For each $f \in C_c(H)$ and $x, y \in H$, one may define $L_x f(y) := \delta_x * \delta_y(f)$. A Borel measure h on H is called a (left) **Haar measure** if $h(L_x f) = h(f)$ for all $f \in C_c(H)$ and $x \in H$. Also

$$f *_h g(x) := \int_H f(t) L_{\tilde{t}} g(x) dh(t), \quad \text{and} \quad \|f\|_1 := \int_H |f(t)| dh(t)$$

for all $f, g \in C_c(H)$ and $x \in H$. Then the set of all h -integrable functions on H forms a Banach algebra, denoted by $(L^1(H, h), *_h, \|\cdot\|_1)$; it is called the **hypergroup algebra** of H .

Let H be a hypergroup with a Haar measure h . Then the **modular function** Δ is defined on H by the identity $h * \delta_{\tilde{x}} = \Delta(x)h$ for every $x \in H$. Δ is continuous, and over every set $\{x\} * \{y\}$, for all $x, y \in H$, Δ is constantly equal to $\Delta(x)\Delta(y)$. In particular, if $\Delta(x)\Delta(\tilde{x}) = \Delta(\delta_x * \delta_{\tilde{x}}) = \Delta(e) = 1$. Also similar to locally compact groups,

$$\int_H f(\tilde{y}) dh(y) = \int_H \frac{1}{\Delta(y)} f(y) dh(y). \quad (1.1)$$

Hypergroups which are from either of compact, discrete, or commutative type, are unimodular i.e. $\Delta \equiv 1$ on H .

For some classes of hypergroups including discrete and/or commutative and/or compact hypergroups, the existence of a Haar measure can be proved. So from now on, we assume that every hypergroup H possesses a Haar measure. Note that unlike groups, the Haar measure on discrete hypergroups is not necessarily a fixed multiplier of the counting measure. Let H be a discrete hypergroup equipped with a Haar measure h . Then, $\ell^1(H) = M(H)$ is a Banach algebra. Also the mapping $f \mapsto fh$, $L^1(H, h) \rightarrow \ell^1(H)$ is an isometric algebra isomorphism from the Banach algebra $L^1(H, h)$ onto the Banach algebra $\ell^1(H)$. The following proposition is a discrete version of [7, Proposition 1.2.16]. Here we present a short proof applying the definition of discrete hypergroups.

Proposition 1.2 *Let H be a discrete hypergroup. Then for every $\phi \in c_0(H)$ and $f \in L^1(H, h)$, the function $f * \phi$ belongs to $c_0(H)$.*

Proof. Let $\epsilon > 0$ be fixed. Therefore there is some $K \subset H$ finite such that for every $x \in H \setminus K$, $|\phi(x)| < \epsilon \|f\|_1^{-1}$. Also there is some $F \subseteq H$ finite such that

$$\sum_{x \in H \setminus F} |f(x)|h(x) < \epsilon \|\phi\|_\infty^{-1}.$$

Based on the definition of convolution between sets and (H1) in Definition 1.1, it is obvious that $C := F * K$ is a finite subset of H . Let $x \in H \setminus C$, $t \in F$, and $s \in K$. If $\delta_{\tilde{t}} * \delta_x(s) \neq 0$, $s \in \tilde{t} * x$. Therefore, by (H6), $e \in \tilde{s} * \tilde{t} * x$. Again (H6) implies that $\tilde{x} \in \tilde{s} * \tilde{t}$ or equivalently $x \in t * s \subseteq F * K$ which is a contradiction. Hence, for $x \in H \setminus C$, $t \in F$, and $s \in K$, $\delta_{\tilde{t}} * \delta_x(s) = 0$. Consequently,

$$\sum_{t \in F} |f(t)| \sum_{s \in K} |\phi(s)| \delta_{\tilde{t}} * \delta_x(s) h(t) = 0.$$

Therefore for $x \in H \setminus C$, one gets

$$\begin{aligned} \left| \sum_{t \in H} f(t) \phi(\delta_{\tilde{t}} * \delta_x) h(t) \right| &\leq \left| \sum_{t \in F} f(t) \phi(\delta_{\tilde{t}} * \delta_x) h(t) \right| + \left| \sum_{t \in H \setminus F} f(t) \phi(\delta_{\tilde{t}} * \delta_x) h(t) \right| \\ &\leq \sum_{t \in F} |f(t)| |\phi(\delta_{\tilde{t}} * \delta_x)| h(t) + \sum_{t \in H \setminus F} |f(t)| h(t) \|\phi\|_\infty \\ &\leq \epsilon + \sum_{t \in F} |f(t)| \sum_{s \in H} |\phi(s)| \delta_{\tilde{t}} * \delta_x(s) h(t) \\ &= \epsilon + \sum_{t \in F} |f(t)| \sum_{s \in H \setminus K} |\phi(s)| \delta_{\tilde{t}} * \delta_x(s) h(t) \\ &\quad + \sum_{t \in F} |f(t)| \sum_{s \in K} |\phi(s)| \delta_{\tilde{t}} * \delta_x(s) h(t) \\ &\leq \epsilon + \sup_{s \in H \setminus K} |\phi(s)| \|f\|_1 + \sum_{t \in F} |f(t)| \sum_{s \in K} |\phi(s)| \delta_{\tilde{t}} * \delta_x(s) h(t) = 2\epsilon. \end{aligned}$$

And this finishes the proof. \square

Let $C(H)$ denote the set of all continuous bounded functions on H . If H is a commutative hypergroup, the **dual of H** , denoted by \widehat{H} , is defined to be the set

$$\{\alpha \in C(H) \mid \alpha(\delta_x * \delta_y) = \alpha(x)\alpha(y), \alpha(\tilde{x}) = \overline{\alpha(x)}\}.$$

One can show that \widehat{H} is the set of all $*$ -multiplicative functionals of $L^1(H, h)$. Therefore, \widehat{H} can be equipped with the Gelfand spectrum topology as the maximal ideal space of the hypergroup algebra which forms a locally compact space. For each $f \in L^1(H)$, the Gelfand transform $f \mapsto \mathcal{F}f$ is defined by

$$\mathcal{F}f(\alpha) = \int_H f(x) \overline{\alpha(x)} dh(x) \quad (\alpha \in \widehat{H})$$

and is called **Fourier transform** which is a norm decreasing injection from $L^1(H, h)$ into $C_0(\widehat{H})$. Also one may define the **Fourier-Stieltjes transform** $\mathcal{FS} : M(H) \rightarrow C(\widehat{H})$ as another norm decreasing injection by

$$\mathcal{FS}(\mu)(\alpha) := \int_H \overline{\alpha(x)} d\mu(x) \quad (\alpha \in \widehat{H}).$$

There exists a measure ϖ on \widehat{H} such that for every $f \in L^1(H, h) \cap L^2(H, h)$,

$$\int_H |f(x)|^2 dh(x) = \int_{\widehat{H}} |\mathcal{F}(f)(\alpha)|^2 d\varpi(\alpha).$$

The measure ϖ is called the **Plancherel measure**.

1.2 Fourier algebra of hypergroups

For a compact hypergroup H , Vrem in [43] defined the **Fourier space** similar to the Fourier algebra of a compact group. Subsequently, Muruganandam, [34], defined the **Fourier-Stieltjes space** on an arbitrary (not necessary compact) hypergroup H using irreducible representations of H as analogous to the Fourier-Stieltjes algebra on locally compact groups. Subsequently, he defined the **Fourier space** of a hypergroup H , as a closed subspace of the Fourier-Stieltjes algebra, generated by $\{f *_h \tilde{f} : f \in L^2(H, h)\}$ or equivalently generated by $\{f *_h \tilde{f} : f \in C_c(H)\}$; hence, $A(H) \cap C_c(H)$ is dense in $A(H)$. Further, $A(H) \subseteq C_0(H)$, $\|\cdot\|_\infty \leq \|\cdot\|_{A(H)}$, and for every $u \in A(H)$, $L_x u$, \tilde{u} , and \bar{u} belong to $A(H)$.

For a hypergroup H , it is known that for every $x \in H$ and $f \in L^2(H)$, $L_x f \in L^2(H)$ while $\|L_x f\|_2 = \|L_x\|_2$ (see [7, (1.3.18)]). Therefore, L_x is an operator in $\mathcal{B}(L^2(H))$ which is denoted by $\lambda(x)$. The von Neumann sub-algebra of $\mathcal{B}(L^2(H))$ generated by $(\lambda(x))_{x \in H}$ is called the **hypergroup von Neumann algebra** of H and denoted by $VN(H)$. By [34, Theorem 2.19], for every $T \in VN(H)$ there exists a unique continuous linear functional ϕ_T on $A(H)$ satisfying $\phi_T(u) = \langle T(f), g \rangle_{L^2(H)}$ where $\tilde{u} = f *_h \tilde{g}$. The mapping $T \mapsto \phi_T$ is a Banach space isomorphism between $VN(H)$ and $A(H)^*$. Moreover, the above mapping is also a homeomorphism when $VN(H)$ is given the σ -weak topology and $A(H)^*$ is given weak* topology.

On the other hand, for each $f \in L^1(H)$, $f *_h g \in L^2(H)$ for $g \in L^2(H)$ while $\|f *_h g\|_2 \leq \|f\|_1 \|g\|_2$. So the operator $\lambda(f)$ which carries g to $f *_h g$ belongs to $\mathcal{B}(L^2(H))$. The C^* -algebra generated by $(\lambda(f))_{f \in L^1(H)}$ in $\mathcal{B}(L^2(H))$ is called **reduced C^* -algebra** of H and denoted by $C_\lambda^*(H)$. It is proven in [34] that $C_\lambda^*(H)$ is actually a C^* -subalgebra of $VN(H)$. Moreover, $A(H)$ can be considered as a subalgebra of $B_\lambda(H)$ where $B_\lambda(H)$ is the dual of $C_\lambda^*(H)$. In this paper we rely on the following lemma which we present from [1] without its proof.

Lemma 1.3 [1, Lemma 3.4]

Let H be a hypergroup, K a compact subset of H and U an open subset of H such that $K \subset U$. Then for each relatively compact open set V such that $\overline{K * V * \check{V}} \subseteq U$, then $u_V := h_H(V)^{-1} 1_{K*V} *_h \check{1}_V$ belongs to $A(H) \cap C_c(H)$. Also $u_V(H) \geq 0$, $u_V|_K = 1$, $\text{supp}(u_V) \subseteq U$, and

$$\|u_V\|_{A(H)} \leq \left(\frac{h_H(K * V)}{h_H(V)} \right)^{\frac{1}{2}}.$$

Remark 1.4 For each pair K, U such that $K \subset U$, we can always find a relatively compact neighborhood V of e_H that satisfies the conditions in Lemma 1.3. The existence is a result of continuity of the mapping $(x, y) \mapsto x * y$ with respect to the locally compact topology of $H \times H$ into the Michael topology on $\mathfrak{L}(H)$ (see [7]). Since H is locally compact, there exists some relatively compact open set W such that $K \subseteq W \subseteq \overline{W} \subseteq U$; $K \in \mathfrak{L}_{H \setminus \overline{W}}(W)$ as an open set in the Michael topology and consequently for each $x \in K$, $x * e \in \mathfrak{L}_{H \setminus \overline{W}}(W)$. Since, the mapping $e \rightarrow x * e$ is continuous, there is some neighborhood V_1^x of e such that for each $y \in V_1^x$, $x * y \in \mathfrak{L}_{H \setminus \overline{W}}(W)$ i.e. $x * y \subseteq W$ and $x * y \cap H \setminus \overline{W} = \emptyset$. Let us define $V^{(1)} = \cup_{x \in K} (V_1^x \cap \check{V}_1^x)$. Clearly, $\check{V}^{(1)} = V^{(1)}$. Moreover, $K * V^{(1)} = \cup_{y \in V^{(1)}} \cup_{x \in K} x * y \subseteq \cup_{x \in K} x * V_1^x \subseteq W$ and $K * V^{(1)} \cap H \setminus \overline{W} = \emptyset$ since $(x * y) \cap (H \setminus \overline{W}) = \emptyset$ for all $x \in K$ and $y \in V^{(1)}$. Now let us replace K by the compact set $\overline{K * V^{(1)}}$. Therefore, similar to the previous argument, for some relatively compact open set W' such that $\overline{K * V^{(1)}} \subseteq W' \subseteq \overline{W'} \subseteq U$, one may find some $V^{(2)}$ a neighborhood of e such that $V^{(2)} = \check{V}^{(2)}$, $\overline{K * V^{(1)}} * V^{(2)} \subseteq W'$, and $(\overline{K * V^{(1)}} * V^{(2)}) \cap (H \setminus \overline{W'}) = \emptyset$. Hence, for the relatively compact open set $V := V^{(1)} \cap V^{(2)}$, one gets that $V = \check{V}$ and $K * V * \check{V} \subseteq \overline{K * V^{(1)}} * V^{(2)} \subseteq \overline{W'}$. So $\overline{K * V * \check{V}} \subseteq U$.

In [34], Muruganandam showed that when H is commutative, $A(H)$ can be characterized as $\{f *_h \tilde{g} : f, g \in L^2(H, h)\}$ and $\|u\|_{A(H)} = \inf \|f\|_2 \|g\|_2$ for all $f, g \in L^2(H, h)$ such that $u = f *_h \tilde{g}$. The key point of his proof for commutative hypergroups, as it was shown in [34, Proposition 4.2] and [11, Section 2], is this fact that $\mathcal{F}(A(H))$, where \mathcal{F} is the (extension of the) Fourier transform, is $L^1(S, \varpi)$ where S , as a subset of \widehat{H} , is the support of the Plancherel measure ϖ . A similar characterization is known for the Fourier space of compact hypergroups, [43]. The following implies that this fact is true for all hypergroups.

Proposition 1.5 Let H be a hypergroup. Then $A(H) = \{f * \tilde{g} : f, g \in L^2(H)\}$ and $\|u\| = \inf \{\|f\|_2 \|g\|_2\}$ over all $f, g \in L^2(G)$ where $u = f * \tilde{g}$. This infimum is actually attained for some $f, g \in L^2(G)$.

As a great reference for this important characterization for locally compact groups is the Master's thesis of Zwarich, [45]. Chapter 4 in this long thesis is dedicated to this proof for

locally compact groups based on an observation by Haagerup in [16] and the von Neumann theory developed in [13]. The proof, in [45], is written based on the properties of von Neumann algebras which are in **standard form** as defined in [16], so it can be adapted easily for hypergroups as well. Before we present the proof, let us recall that every element $u \in A(H)$ acts σ -weak continuously on $VN(H)$; hence, u is actually a **normal** linear functional on $VN(H)$, [45, Section 3.3].

Proof. For $f, g \in C_c(H)$, define

$$\langle f, g \rangle := \int_H f(x) \overline{g(x)} dx, \quad f^*(x) := \frac{\overline{f(\tilde{x})}}{\sqrt{\Delta(x)}}, \quad f^\sharp(x) := \frac{1}{\sqrt{\Delta(x)}} f(x).$$

Then $C_c(H)$ forms a **quasi-Hilbert algebra** as defined in [45, Definition 4.3.3]. To show that, one can re-write the proof of [45, Proposition 4.3.4] with appropriate modifications and applying (1.1). Obviously the Hilbert space generated by $C_c(H)$ would be $L^2(H)$. Further, the second commutant of $\lambda(C_c(H))$ is $VN(H)$, the von Neumann algebra of H (see [34, Remark 2.18]). Therefore, by [45, Theorem 4.3.9], $VN(H)$ is in standard form. Hence, one may apply [45, Theorem 4.3.16] to the elements of $A(H)$ as normal linear functionals on $VN(H)$ and conclude that $A(H) = \{f * \tilde{g} : f, g \in L^2(H)\}$ and the condition of the norm. \square

In [34], Muruganandam calls a hypergroup H a **regular Fourier hypergroup**, if the Banach space $(A(H), \|\cdot\|_{A(H)})$ equipped with pointwise product is a Banach algebra. He studied this property for a variety of commutative hypergroups in [34]. He showed that some polynomial hypergroups including Jacobi polynomial hypergroups and Chebyshev polynomial hypergroups are regular Fourier hypergroups. Furthermore, in [35], he pursued this study for **double coset hypergroups** (which are not necessarily commutative). He showed that the **ultraspherical hypergroups**, including the double coset hypergroups, are regular Fourier. For a compact group G , the equivalent classes of irreducible unitary representations forms a discrete commutative hypergroup denoted by \widehat{G} and called **dual of G** , (see Subsection 2.2). In [1], it was shown dual of every compact group is a regular Fourier hypergroup. Moreover, the Fourier algebra $A(\widehat{G})$ as a Banach algebra is isometrically isomorphic to the center of the group algebra usually denoted by $ZL^1(G)$.

2 Følner type conditions on Hypergroups

Amenable locally compact groups are characterized by a variety of properties including Følner type conditions. As we mentioned before, these conditions relate the concept of “amenability” (which is an algebraic notion on the group algebra) to some structural properties of the group or semigroup. In this section, we look at a generalization of Følner type conditions over hypergroups.

2.1 Definitions and relations

In [1], the author introduced the Leptin condition for hypergroups. Here, we define more Følner type conditions for hypergroups and we study their relations. To recall, for each two subsets A and B of some set X , we denote their symmetric difference, $(A \setminus B) \cup (B \setminus A)$, by $A \triangle B$.

Definition 2.1 Let H be a hypergroup and $D \geq 1$ an integer. We define the following properties:

- (L_D) We say that H satisfies the *D-Leptin condition* if for every compact subset K of H and $\epsilon > 0$, there exists a measurable set V in H such that $0 < h(V) < \infty$ and $h(K * V)/h(V) < D + \epsilon$.
- (F) We say that H satisfies the *Følner condition* if for every compact subset K of H and $\epsilon > 0$, there exists a measurable set V in H such that $0 < h(V) < \infty$ and $h(x * V \triangle V)/h(V) < \epsilon$ for every $x \in K$.
- (SF) We say that H satisfies the *Strong Følner condition* if for every compact subset K of H and $\epsilon > 0$, there exists a measurable set V in H such that $0 < h(V) < \infty$ and $h(K * V \triangle V)/h(V) < \epsilon$.

Remark 2.2 If a hypergroup H satisfies the 1-Leptin condition, H satisfies the *Leptin condition* as defined in [1, Definition 4.1]. From now on, we may use the Leptin condition instead of the 1-Leptin condition and we denote it by (L) .

Proposition 2.3 *For every compact hypergroup H , H satisfies all conditions (SF) , (F) , and (L) .*

Proof. The proof is a direct result of finiteness of the Haar measure on compact hypergroups, [7], by replacing $V = H$ for all conditions in Definition 2.1. \square

Remark 2.4 In Definition 2.1 of the Leptin condition, (L_D) , we can suppose that V is compact. To show this fact suppose that H satisfies the D -Leptin condition. For compact subset K of H and $\epsilon > 0$, there exists a measurable set V such that $h(K * V)/h(V) < D + \epsilon$. Using regularity of h , as a measure, for each positive integer n , we can find compact set $V_1 \subseteq V$ such that $h(V \setminus V_1) < h(V)/n$. This implies that $0 < h(V_1)$ and $h(V)/h(V_1) < n/(n-1)$. Therefore

$$\frac{h(K * V_1)}{h(V_1)} \leq \frac{h(V)}{h(V_1)} \left(\frac{h(K * V_1)}{h(V)} \right) < \frac{n}{n-1} (D + \epsilon).$$

So we can add compactness of V to the definition of the Leptin condition.

Proposition 2.5 *For every hypergroup H , (SF) implies (L) .*

Proof. For a compact set K and $\epsilon > 0$, let V be a measurable set such that $h(K * V \triangle V) < \epsilon h(V)$. Hence

$$\begin{aligned} \frac{h(K * V)}{h(V)} - 1 &\leq \frac{h(K * V) - h(V)}{h(V)} \\ &\leq \frac{h(K * V) + h(V) - 2h((K * V) \cap V)}{h(V)} \\ &= \frac{h((K * V) \triangle V)}{h(V)} < \epsilon. \end{aligned}$$

□

Proposition 2.6 *For every discrete hypergroup H , (F) implies (SF) . And consequently, (F) implies (L) .*

Proof. We should just show that $(F) \Rightarrow (SF)$ the rest is obtained by Proposition 2.5. Let K be a compact subset of H . Since for discrete hypergroups, each compact set is finite, we may suppose that $K = \{x_i\}_{i=1}^n$. Therefore, for each $\epsilon > 0$ there is a finite set V such that $0 < h(V)$ and

$$\frac{h((x * V) \triangle V)}{h(V)} < \frac{\epsilon}{|K|} \quad (x \in K).$$

So

$$\begin{aligned} \frac{h((\bigcup_{i=1}^n x_i) * V \triangle V)}{h(V)} &= \frac{h(\bigcup_{i=1}^n (x_i * V) \triangle V)}{h(V)} \\ &\leq \sum_{i=1}^n \frac{h(x_i * V \triangle V)}{h(V)} = \epsilon. \end{aligned}$$

The last inequality is a result of the following inclusion about arbitrary sets B_1, B_2, C :

$$((B_1 \cup B_2) \triangle C) \subseteq (B_1 \triangle C) \cup (B_2 \triangle C).$$

□

Remark 2.7 If H is a locally compact group, all the conditions (F) , (SF) , and (L) are equivalent and they equal the amenability of the group H . If one tries to adapt the rest of the relations between (F) , (SF) , and (L) from the group case, [36], one may notice that in almost all of the arguments, the inclusion $x(A \setminus B) \subseteq xA \setminus xB$ is crucially applied where A, B are subsets of the group H and x is one arbitrary element.¹ This inclusion is not necessarily true for a general hypergroup though.

¹Note that in general groups the equality holds, but this side of the inclusion suffices.

In [20] the authors rendered the notion of Følner conditions on polynomial hypergroups. To do so, summing sequences in the context of polynomial hypergroups are defined as follows.

Definition 2.8 [20, Definition 2.1]

Let \mathbb{N}_0 denote a polynomial hypergroup with the Haar measure h . A sequence $(A_n)_{n \in \mathbb{N}_0}$ where $A_n \subseteq \mathbb{N}_0$ for all $n \in \mathbb{N}$ is called *summing sequence* on the polynomial hypergroup \mathbb{N}_0 if it satisfies

- (1) $A_n \subseteq A_{n+1}$ for every $n \in \mathbb{N}_0$,
- (2) $\mathbb{N}_0 = \bigcup_{n \in \mathbb{N}_0} A_n$,
- (3) $h(A_n) < \infty$ for every $n \in \mathbb{N}_0$,
- (4) $\lim_{n \rightarrow \infty} \frac{h((k * A_n) \Delta A_n)}{h(A_n)} = 0$ for all $k \in \mathbb{N}$.

Example 2.9 Let \mathbb{N}_0 be a polynomial hypergroup which have a summing sequence $(A_n)_{n \in \mathbb{N}_0}$. Then it satisfies all the Leptin, Strong Følner, and Følner conditions. To prove this, note that the existence of a summing sequence immediately implies the Følner condition, the rest would be proven based on Proposition 2.6, since \mathbb{N}_0 is a discrete commutative hypergroup. As an example in [20], it was shown that Jacobi polynomials have summing sequences.

2.2 D -Leptin condition on dual of compact groups

Let G be a compact group and \widehat{G} the set of all equivalent classes of irreducible unitary representations of G . When \mathcal{H}_π is the finite dimensional Hilbert space related to a representation $\pi \in \widehat{G}$, d_π denotes the dimension of the \mathcal{H}_π .

For each two irreducible representations $\pi_1, \pi_2 \in \widehat{G}$, $\pi_1 \otimes \pi_2$ can be written as a decomposition of π'_1, \dots, π'_n elements of \widehat{G} with respective multiplicities m_1, \dots, m_n , i.e. $\pi_1 \otimes \pi_2 \cong \bigoplus_{i=1}^n m_i \pi'_i$. According to this decomposition, one may define a convolution and involution on \widehat{G} to $\ell^1(\widehat{G})$ by

$$\delta_{\pi_1} * \delta_{\pi_2} := \sum_{i=1}^n \frac{m_i d_{\pi'_i}}{d_{\pi_1} d_{\pi_2}} \delta_{\pi'_i} \quad \text{and} \quad \tilde{\pi} = \bar{\pi} \quad (2.1)$$

for all $\pi, \pi_1, \pi_2 \in \widehat{G}$ where $\bar{\pi}$ is the complex conjugate of the representation π . Then $(\widehat{G}, *, \tilde{\cdot})$ forms a discrete commutative hypergroup such that π_0 , the trivial representation of G , is the identity element of \widehat{G} and $h(\pi) = d_\pi^2$ is the Haar measure of \widehat{G} .

Corollary 2.10 [1, Section 4]

The hypergroups \widehat{G} satisfies the Følner, strong Følner, and Leptin conditions for $G = \text{SU}(2)$ or $G = \prod_{i \in \mathbf{I}} G_i$ the product equipped with product topology for $\{G_i\}_{i \in \mathbf{I}}$ a family of finite groups.

In [1], it was implied that the duals of compact groups, as discrete commutative hypergroups, are regular Fourier hypergroups. This fact was applied to study some properties of compact groups, using the Fourier algebra of the dual of compact groups. That study mainly was based on the satisfaction of Leptin condition by the dual hypergroups.

Let \mathbb{G} be a connected simply connected compact real Lie group, (e.g. $SU(n)$). Then, $\widehat{\mathbb{G}}$, as the dual object of a compact Lie groups, forms a finitely generated hypergroup. Suppose that F is a finite generator of $\widehat{\mathbb{G}}$; therefore, by [6, Theorem 2.1], there exists positive integers $0 < \alpha, \beta < \infty$ such that

$$\alpha \leq \frac{h_{\widehat{\mathbb{G}}}(F^k)}{k^{d_{\mathbb{G}}}} \leq \beta \quad (2.2)$$

for all $k \in \mathbb{N}$ where $d_{\mathbb{G}}$ is the dimension of the group \mathbb{G} as a Lie group over \mathbb{R} . According to the following theorem, this estimation for the growth rate of $\widehat{\mathbb{G}}$ results in the satisfaction of D -Leptin condition for $\widehat{\mathbb{G}}$.

Theorem 2.11 *Let \mathbb{G} be a connected simply connected compact real Lie group. Then $\widehat{\mathbb{G}}$, as a hypergroup, satisfies the D -Leptin condition for some $D \geq 1$.*

Proof. Given finite set $K \subseteq \widehat{\mathbb{G}}$. Suppose that F is a finite generator of $\widehat{\mathbb{G}}$. For some $k \in \mathbb{N}$, $K \subseteq F^k$. Moreover, for each $\ell \in \mathbb{N}$, $F^\ell * F^k \subseteq F^{\ell+k}$. By applying (2.2),

$$\limsup_{\ell \rightarrow \infty} \frac{h_{\widehat{\mathbb{G}}}(K * F^\ell)}{h_{\widehat{\mathbb{G}}}(F^\ell)} \leq \limsup_{\ell \rightarrow \infty} \frac{h_{\widehat{\mathbb{G}}}(F^{\ell+k})}{h_{\widehat{\mathbb{G}}}(F^\ell)} = \limsup_{\ell \rightarrow \infty} \frac{h_{\mathbb{G}}(F^{\ell+k})}{(\ell+k)^{d_{\mathbb{G}}}} \frac{\ell^{d_{\mathbb{G}}}}{h_{\widehat{\mathbb{G}}}(F^\ell)} \frac{(\ell+k)^{d_{\mathbb{G}}}}{\ell^{d_{\mathbb{G}}}} \leq \beta/\alpha.$$

Therefore, $\widehat{\mathbb{G}}$ satisfies the D -Leptin condition for some $1 \leq D < \infty$. \square

Let $SU(3)$ denote the special group of 3×3 unitary matrices which is a connected simply connected compact real Lie group. Here we apply some studies on the representation theory of real connected Lie groups to find a concrete answer for D for this special hypergroup.

Proposition 2.12 *The hypergroup $\widehat{SU(3)}$ satisfies the 18240-Leptin condition.*

Proof. Let us follow [8] in notations and basic facts about a compact Lie group \mathbb{G} and in particular $\mathbb{G} = SU(3)$. Let the set of all fundamental weights β be denoted by B . Then we have $(\beta|\beta') = 0$ for $\beta \neq \beta'$ while $(\beta|\beta) > 0$ for all $\beta, \beta' \in B$. Taking highest weights induces an identification between the set $\widehat{\mathbb{G}}$ of classes of irreducible unitary representations and the set of dominant weights X_{++} which are all $(p_\beta\beta)_\beta$ for $p_\beta \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$. From now on, without loss of generality, we denote the element π of $\widehat{\mathbb{G}}$ by its corresponding multipliers in X_{++} that is $\pi = (p_\beta)_\beta$. As it is mentioned in [6], we know that in the case of connected simply connected compact real Lie groups, the set F , of representations δ_{β_0} which is $\delta_{\beta_0} = (p_\beta)_\beta$ where $p_\beta = 0$ for

all $\beta \neq \beta_0$ and $p_{\beta_0} = 1$, forms a generator of $\widehat{\mathbb{G}}$. Further, one may define a mapping $\tau : \widehat{\mathbb{G}} \rightarrow \mathbb{N}_0$ where $\tau(\pi) = \sum_{\beta} p_{\beta}$ such that for each $\pi = (p_{\beta})_{\beta}$, π belongs to $F^{\tau(\pi)} \setminus F^{\tau(\pi)-1}$. The dimension of $\pi = (p_{\beta})_{\beta}$ is given by Weyl's formula

$$d_{\pi} = \prod_{\alpha \in R^+} \left(1 + \frac{\sum_{\beta} p_{\beta}(\beta|\alpha)}{(\rho|\alpha)} \right)$$

where ρ is the sum of the fundamental dominant weights where $(\rho, \beta) = 1$ for each β .

If we restrict the case to $\text{SU}(3)$, one gets that $\widehat{\text{SU}(3)}$ is nothing but the set of all $\pi = (p, q)$ where $p, q \in \mathbb{N}_0$ while $d_{\pi} = (p+1)(q+1)(2+p+q)/2$. Also according to the finite generator $F = \{(0, 1), (1, 0)\}$, one gets that $S_k := F^k \setminus F^{k-1}$ is nothing but the set $\{(k, k-j)\}_{j=0}^k$. Hence, based on some computations, one gets that

$$h(S_k) = \sum_{j=0}^k \frac{(j+1)^2(k-j+1)^2(k+2)^2}{4}.$$

One may use this fact that $F^k = \sum_{j=0}^k S_j$ to get the following bounds. Therefore,

$$\frac{h(F^n)}{n^8} = \frac{1}{n^8} + \frac{3n^7 + 60n^6 + 518n^5 + 2520n^4 + 7547n^3 + 14220n^2 + 16412n + 10560}{2880n^7}.$$

Therefore,

$$\frac{1}{960} < \frac{h(F^n)}{k^n} \leq 19. \quad (2.3)$$

Now the argument mentioned in the proof of Theorem 2.11 implies that $\widehat{\text{SU}(3)}$ satisfies the 18240-Leptin condition. \square

Remark 2.13 In [2], the author applied a study on the tensor decomposition of irreducible representations of $\text{SU}(3)$, [44], to compute the D -Leptin condition of the dual of $\text{SU}(3)$. The outcome was the 3^8 -Leptin condition which is significantly smaller than the amount found in in Proposition 2.12. But still the advantage of the proof of Proposition 2.12 is that the structural details in the first half of the proof for Proposition 2.12 let us to have similar computations for other $\text{SU}(n)$'s and find two upper and lower bounds α and β , although these doable computations would be really long for large n 's (as even it is for $n = 3$). It sounds to be interesting if one can apply this theory to obtain a formula for $\text{SU}(n)$ or even for other connected simply connected real compact Lie groups. To do so, one may find the computations in the proof of [6, Theorem 2.1] helpful. Note that the real dimension of the group $\text{SU}(3)$ is 8; hence, $\alpha = 1/960$ and $\beta = 19$ are actually the bounds which are mentioned in (2.2).

Suppose that $\{G_i\}_{i \in \mathbf{I}}$ is a non-empty family of compact groups for an arbitrary indexing set \mathbf{I} . Let $G := \prod_{i \in \mathbf{I}} G_i$ be the product of $\{G_i\}_{i \in \mathbf{I}}$ equipped with product topology. Then G is a compact group and by [18, Theorem 27.43], \widehat{G} is the discrete space of all $\pi = \otimes_{i \in \mathbf{I}} \pi_i$ such that every π_i belongs to \widehat{G}_i and π_i is the trivial representation 1 on G_i for all except for finitely many $i \in \mathbf{I}$. Moreover, for each $\pi = \otimes_{i \in \mathbf{I}} \pi_i \in \widehat{G}$, $d_\pi = \prod_{i \in \mathbf{I}} d_{\pi_i}$.

Theorem 2.14 *Let $G = \prod_{i \in \mathbf{I}} G_i$ for a family of compact groups $(G_i)_{i \in \mathbf{I}}$ such that for each $i \in \mathbf{I}$, \widehat{G}_i satisfies the D_i -Leptin condition. Then if $D := \prod_{i \in \mathbf{I}} D_i$ exists, \widehat{G} satisfies the D -Leptin condition.*

Proof. Given compact subset K of \widehat{G} and $\epsilon > 0$. There exists some finite set $F \subseteq \mathbf{I}$ such that $K \subseteq \bigotimes_{i \in F} K_i \otimes E_F^c$ where K_i is a compact subset of \widehat{G}_i and $E_F^c = \bigotimes_{i \in \mathbf{I} \setminus F} \pi_0$ where π_0 's are the identities of the corresponding hypergroup \widehat{G}_i . If $D := \prod_{i \in \mathbf{I}} D_i < \infty$, given $\epsilon > 0$, one may find a $\epsilon' > 0$ such that $\prod_{i \in F} (D_i + \epsilon') < D + \epsilon$. Using the D_i -Leptin condition for each \widehat{G}_i , there exists some finite set V_i which satisfies such that $h_{\widehat{G}_i}(K_i * V_i) / h_{\widehat{G}_i}(V_i) < D_i + \epsilon'$. Therefore, for the finite set $V = (\bigotimes_{i \in F} V_i) \otimes E_F^c$,

$$\frac{h(K * V)}{h(V)} \leq \prod_{i \in F} \frac{h_{G_i}(K_i * V_i)}{h_{G_i}(V_i)} < \prod_{i \in F} (D_i + \epsilon') < D + \epsilon.$$

□

3 Bounded approximate identity of Fourier algebra

Let H be a regular Fourier hypergroup, we denote by (B_D) the existence of an approximate identity of $A(H)$ whose $\|\cdot\|_{A(H)}$ -norm is bounded by some $D \geq 1$ and we call such a bounded approximate identity a ***D*-bounded approximate identity**. If H is a locally compact group, it is known that $A(H)$ has a 1-bounded approximate identity if and only if H is amenable. Here we study the existence of a bounded approximate identity of $A(H)$ with respect to the properties of the hypergroup H .

Theorem 3.1 *Let H be a regular Fourier hypergroup which satisfies the D -Leptin condition, (L_D) , for some $D \geq 1$. Then $A(H)$ has a D -bounded approximate identity, (B_D) .*

Proof. Fix $\epsilon > 0$. Using the D -Leptin condition on H , for every arbitrary non-void compact set K in H , we can find a finite subset V_K of H such that $h(K * V_K) / h(V_K) < D^2(1 + \epsilon)^2$. Using Lemma 1.3, for

$$v_K := \frac{1}{h(V_K)} 1_{K * V_K} *_h \tilde{1}_{V_K}$$

we have $\|v_K\|_{A(H)} < D(1 + \epsilon)$ and $v_K|_K \equiv 1$. Define for each pair (K, ϵ) , $a_{\epsilon, K} = (1 + \epsilon)^{-1}v_K$. We consider the net $\{a_{\epsilon, K} : K \subseteq H \text{ compact, and } 0 < \epsilon < 1\}$ in $A(H)$ where $a_{\epsilon_1, K_1} \preceq a_{\epsilon_2, K_2}$ whenever $v_{K_1}v_{K_2} = v_{K_1}$ and $\epsilon_2 < \epsilon_1$. So $(a_{\epsilon, K})_{0 < \epsilon < 1, K \subseteq H}$ forms a $\|\cdot\|_{A(H)}$ -norm D -bounded net in $A(H) \cap C_c(H)$. Let $f \in A(H) \cap C_c(H)$ with $K = \text{supp } f$. Then $v_K f = f$. Therefore, $(a_{\epsilon, K})_{0 < \epsilon < 1, K \subseteq H}$ is a D -bounded approximate identity of $A(H)$. \square

Remark 3.2 Note that if H is a locally compact group such that $A(H)$ has a D -bounded approximate identity, one may prove that H satisfies the (D) -Leptin condition. But an argument similar to the group one cannot be applied to the hypergroup case, since for every measurable set $E \subseteq H$ and $x \in H$, $L_x 1_E \neq 1_{\tilde{x}*E}$ necessarily when H is a hypergroup and 1_E denotes the character function on E . It is interesting if one can find a regular Fourier hypergroup with (B_D) which does not satisfy (L_D) .

Proposition 3.3 *Let H be a commutative hypergroup such that \widehat{H} , as the dual of H , has hypergroup structure. Then H is a regular Fourier hypergroup and $A(H)$ has a 1-bounded approximate identity.*

Proof. This proposition based on this fact that $A(H)$ is isometrically isomorphic to the hypergroup algebra $L^1(\widehat{H})$ through the Fourier transform. Also it is known that hypergroup algebras have 1-bounded approximate identities and this finishes the proof. \square

Example 3.4 **Bessel-Kingman hypergroups** are one-dimensional commutative regular Fourier hypergroups which are self dual i.e. $\widehat{H} = H$. So, the Fourier algebra has a 1-bounded approximate identity. See [34] and [7, Section 3.5.61].

Example 3.5 Let G be a compact group. Then the Fourier algebra of \widehat{G} , $A(\widehat{G})$, is isometrically Banach algebra isomorphic to $ZL^1(G)$, [1, Theorem 3.7]. Also since every compact group G is a SIN-group, $ZL^1(G)$ always has a 1-bounded approximate identity. Therefore, $A(\widehat{G})$ has a 1-bounded approximate identity.

3.1 Leptin's Theorem for hypergroups

Definition 3.6 [41, p32]

We say that H satisfies (P_r) , if whenever $\epsilon > 0$ and a compact set $E \subseteq H$ are given, then there exists $f \in L^r(H)$, $f \geq 0$, $\|f\|_r = 1$ such that $\|L_x f - f\|_r < \epsilon$ for every $x \in E$.

We say that H satisfies the **Reiter condition** if it has property (P_1) . For every hypergroup H , (P_1) is equivalent to the amenability of H ; hence, all compact or commutative hypergroups

are satisfying (P_1) . Although (P_2) implies (P_1) , [41], (P_2) is not necessarily equivalent to the amenability of the hypergroup. As a counterexample, one may consider the [Naimark hypergroup](#), see [7, (3.5.66)] and [41, Example 4.6]. One may note that for a commutative hypergroup H with the Plancherel measure ϖ , H satisfies (P_2) if and only if the constant character 1 belongs to $\text{supp}(\varpi)$, [41].

Singh, [40, Proposition 4.4.3], showed that if a hypergroup H satisfies the (1-)Leptin condition, it satisfies (P_r) for any $r \in [1, \infty]$, for r in Definition 3.6.

In the following we crucially rely on [41, Lemma 4.4] which proves that H satisfies (P_2) if and only if there is a net $(f_\alpha)_\alpha \subseteq L^2(H)$ such that $\|f_\alpha\|_2 = 1$ and $f_\alpha * \tilde{f}_\alpha$ converges to 1 uniformly on compact subsets of H . Note that by this lemma, (P_2) implies the existence of a net (g_α) (in the form of $g_\alpha := f_\alpha * \tilde{f}_\alpha$) which belongs to $A(H)$ while $\|g_\alpha\|_{A(H)} \leq \|f_\alpha\|_2^2 = 1$.

The following theorem resembles the Leptin theorem for regular Fourier hypergroups. In the proof, some techniques of the group case (see [39, Theorem 7.1.3]) have been applied.

Recall that a [state](#) on a C^* -algebra is a positive linear functional of norm 1. Moreover, if \mathcal{A} is a von Neumann algebra with predual \mathcal{A}_* , every state of \mathcal{A} can be approximated by a net of states of the elements of pre-dual in the weak* topology. Therefore, for a hypergroup H each state u on $VN(H)$ which belongs to $A(H)$ is in the form of $f *_h \tilde{f}$ for some $f \in L^2(H)$ such that $1 = \|u\|_{A(H)} = u(e) = \|f\|_2^2$, by Proposition 1.5.

Theorem 3.7 *Let H be a regular Fourier hypergroup. Then the following conditions are equivalent.*

- (B_1) $A(H)$ has a 1-bounded approximate identity.
- (B_D) $A(H)$ has a D -bounded approximate identity for some $D \geq 1$.
- (P_2) H satisfies (P_2) .

Proof. $(B_D) \Rightarrow (P_2)$.

For $(e_\alpha)_\alpha$ a D -bounded approximate identity of $A(H)$, there exists a w^* -cluster point $F \in VN(H)^*$. Note that for each $x \in H$, $\langle \lambda(x), F \rangle = \lim_\alpha \langle \lambda(x), e_\alpha \rangle = \lim_\alpha e_\alpha(x) = 1$. So $F|_{L^1(H, h)}$ may be interpreted as the constant function 1 on H (where $L^1(H, h)$ is observed as a subalgebra of $VN(H)$). Therefore, for each $f, g \in L^1(H, h)$, one gets that $\langle F, f * g \rangle = \langle F, f \rangle \langle F, g \rangle$. Hence $F|_{L^1(H, h)}$ is a multiplicative functional on $L^1(H, h)$. Therefore, for each $f \in L^1(H, h)$, $\langle F, \tilde{f} *_h f \rangle = \langle F, \tilde{f} \rangle \langle F, f \rangle = |\langle F, f \rangle|^2 \geq 0$. But $L^1(H, h)$ is dense in the C^* -algebra $C_\lambda^*(H)$; hence, $F|_{C_\lambda^*(H)}$ is a positive functional on $C_\lambda^*(H)$ that is $\langle F, f * \tilde{f} \rangle \geq 0$ for every $f \in C_\lambda^*(H)$. Also as a multiplicative functional, $\|F|_{C_\lambda^*(H)}\| = 1$. But as a positive norm 1 functional, $F|_{C_\lambda^*(H)}$ is a state. Thus, by [32, Corollary 2.3.12], $F|_{C_\lambda^*(H)}$ is extendible to a state E on $VN(H)$. Because states of $VN(H)$ which belong to $A(H)$ are weak* dense in the set of all states of $VN(H)$, we

may find a net $(f_\beta)_\beta$ in $\{f *_h \tilde{f} : f \in L^2(H, h)\}$ such that $f_\beta = g_\beta *_h \tilde{g}_\beta \rightarrow E$ in weak* topology for a net $(g_\beta)_\beta \subseteq L^2(H, h)$. Moreover, $1 = \|f_\beta\|_{A(H)} = f_\beta(e) = g_\beta *_h \tilde{g}_\beta(e) = \|g_\beta\|_2^2$. Since $F|_{C_\lambda^*(H)} = E|_{C_\lambda^*(H)}$, for each $u \in A(H)$ and $f \in L^1(H)$, $uf \in L^1(H)$, we have

$$\lim_\beta \langle uf_\beta, f \rangle = \langle u \cdot E, f \rangle = \langle F, uf \rangle = \lim_\alpha \langle e_\alpha, uf \rangle = \langle u, f \rangle. \quad (3.1)$$

Therefore, $uf_\beta \rightarrow u$ with respect to the topology $\sigma(A(H), L^1(H))$. Recall that $L^1(H)$ is dense in $C_\lambda^*(H)$ while $A(H) \subseteq B_\lambda(H)$ and $B_\lambda(H) = C_\lambda^*(H)^*$. Let us fix $u \in A(H)$. Therefore, for some given $\epsilon > 0$ and $f \in C_\lambda^*(H)$, there is a $g \in L^1(H)$ such that $\|g - f\|_{C_\lambda^*(H)} < \epsilon$. Also there is some β_0 such that for each $\beta \succ \beta_0$, $|\langle uf_\beta - u, g \rangle| < \epsilon$. So,

$$\begin{aligned} |\langle f_\beta u - u, f \rangle| &\leq |\langle f_\beta u - u, f - g \rangle| + |\langle f_\beta u - u, g \rangle| \\ &\leq \|u\|_{A(H)} (\|f_\beta\|_{A(H)} + 1) \|f - g\|_{C_\lambda^*(H)} + \epsilon < (2\|u\|_{A(H)} + 1)\epsilon. \end{aligned}$$

Therefore, $uf_\beta \rightarrow f$ with respect to the topology $\sigma(A(H), C_\lambda^*(H))$ which corresponds to the weak topology on $B_\lambda(H)$. It is a well-known result of functional analysis that the weak closure of a convex set coincides with its norm closure, so for every $\epsilon > 0$, there exists $\varphi_{\{u_1, \dots, u_n\}, \epsilon} = \varphi \in \text{conv}\{f_\beta\}$ such that $u_i \in A(H)$ for $i = 1, \dots, n$ and $\|u_i \varphi - u_i\|_{A(H)} < \epsilon$. Moreover,

$$1 = \varphi(e) \leq \|\varphi\|_\infty \leq \|\varphi\|_{A(H)} \leq 1.$$

Note that φ is also a positive functional in the cone of positive functionals on $VN(H)$; therefore, φ is actually a state and $\varphi = \psi * \tilde{\psi}$ for some $\psi \in L^2(H)$.

To make the set of all such φ 's a net, let $I := \{(S, \epsilon) : S \subseteq A(H) \text{ is finite, } \epsilon > 0\}$ become a directed set by $(S, \epsilon) \leq (S', \epsilon')$ if $S \subseteq S'$ and $\epsilon \geq \epsilon'$. This lets us to render the net $(\varphi_\alpha)_\alpha \subseteq \text{conv}\{f_\beta\}$ that is a bounded approximate identity of $A(H)$. On the other hand, for each compact set $K \subseteq H$, by Lemma 1.3, there is some $u_K \in A(H)$ such that $u_K|_K \equiv 1$. Therefore, for each $x \in K$,

$$\begin{aligned} \lim_\alpha |1 - \varphi_\alpha(x)| &= \lim_\alpha |u_K(x) - u_K(x)\varphi_\alpha(x)| \leq \lim_\alpha \|u_K - u_K\varphi_\alpha\|_\infty \\ &\leq \lim_\alpha \|u_K - u_K\varphi_\alpha\|_{A(H)} = 0. \end{aligned}$$

So $\varphi_\alpha \rightarrow 1$ uniformly on compact subsets of H . Consequently, by [41, Lemma 4.4], the existence of the net $(\varphi_\alpha)_\alpha$ implies (P_2) .

$$(P_2) \Rightarrow (B_1).$$

Let $(g_\beta)_\beta$ be the net generated by (P_2) in [41, Lemma 4.4], that is $g_\beta = f_\beta * \tilde{f}_\beta$ for some $f_\beta \in L^2(H)$ while $\|f_\beta\|_2 = 1$ for every β and $g_\beta \rightarrow 1$ uniformly on compact sets. Therefore,

$$1 = \|f_\beta\|_2^2 = g_\beta(e) \leq \|g_\beta\|_\infty \leq \|g_\beta\|_{A(H)} \leq \|f_\beta\|_2^2 \leq 1.$$

Also for each $u \in A(H) \cap C_c(H)$ and $f \in L^1(H)$,

$$\begin{aligned} \lim_{\beta} |\langle ug_{\beta} - u, f \rangle| &\leq \lim_{\beta} \int_H |u(x)| |g_{\beta}(x) - 1| |f(x)| dx \\ &= \int_{\text{supp}(u)} |u(x)| |g_{\beta}(x) - 1| |f(x)| dx = 0. \end{aligned}$$

Let us fix $u \in A(H)$. For given $\epsilon > 0$ and $f \in L^1(H)$, there is some $v \in A(H) \cap C_c(H)$ such that $\|u - v\|_{A(H)} < \epsilon$ and β_0 such that for any $\beta \succ \beta_0$, $|\langle vg_{\beta} - v, f \rangle| < \epsilon$. So for any $\beta \succ \beta_0$,

$$\begin{aligned} |\langle ug_{\beta} - u, f \rangle| &\leq |\langle ug_{\beta} - vg_{\beta}, f \rangle| + |\langle vg_{\beta} - v, f \rangle| + |\langle v - u, f \rangle| \\ &\leq \|u - v\|_{A(H)} \|g_{\beta}\|_{A(H)} \|f\|_1 + \epsilon + \|v - u\|_{A(H)} \|f\|_1 \\ &< \epsilon(2\|f\|_1 + 1). \end{aligned}$$

Therefore, by one generalization to arbitrary functions on $A(H)$, $\lim_{\beta} ug_{\beta} = u$ in the topology $\sigma(A(H), L^1(H))$. But indeed $A(H) \subseteq B_{\lambda}(H)$ and this topology on bounded subsets of $A(H)$ coincides to the weak topology on $B_{\lambda}(H)$ i.e. $\sigma(B_{\lambda}(H), C_{\lambda}^*(H))$. So similar to the previous part, there is a $(e_{\alpha})_{\alpha} \subset \text{conv}\{g_{\beta}\}_{\beta}$ such that

$$\lim_{\alpha} \|ue_{\alpha} - e_{\alpha}\|_{A(H)} = 0$$

for every $u \in A(H)$. Also note that for each α ,

$$1 = e_{\alpha}(e) \leq \|e_{\alpha}\|_{\infty} \leq \|e_{\alpha}\|_{A(H)} \leq 1.$$

$(B_1) \Rightarrow (B_D)$ is trivial. □

Remark 3.8 Let G be a locally compact group. Then G satisfies the D -Leptin condition for each $D > 1$ if and only if it satisfies the Leptin condition. To observe this fact, note that the existence of a bounded approximate identity for $A(G)$ is equivalent to satisfaction of the Leptin condition by the group G , [39, Theorem 7.1.3].

Example 3.9 Most of the commutative regular Fourier hypergroup examples in [34] including [Jacobi polynomial hypergroups](#), [cosh hypergroup](#), and [generalized Chebyshev polynomials](#) are hypergroups for them the support of the Plancherel measure includes the trivial character; hence, they satisfy (P_2) . Thus the Fourier algebra has a 1-bounded approximate identity.

Let G be a locally compact group G . [Ultraspherical hypergroups](#) are defined in [35] using a linear map $\pi : C_c(G) \rightarrow C_c(G)$ which is called [spherical projector](#) and it satisfies some conditions. This class of hypergroups includes the hypergroup structure defined on the double cosets of

locally compact groups with respect to a compact subgroup. As it was shown in [35], every ultraspherical hypergroup is a regular Fourier hypergroup and the Fourier algebra is isometrically isomorphic to the subalgebra $A_\pi(G) := \{f \in A(G) : f \circ \pi = f\}$ of $A(G)$. Let H be a ultraspherical hypergroup generated by a locally comapet group G . The following corollary is an application of Theorem 3.7 to this class of hypergroups. Recall that before this result we barely knew anything about amenability notions of double coset hypergroups.

Corollary 3.10 *Let H be a ultraspherical hypergroup structure admitted by an amenable locally compact group G . Then H satisfies (P_2) .*

Proof. By [12, Lemma 3.7], the amenability of G implies the existence of a bounded approximate identity for $A(H)$. But as we saw before, H is a regular Fourier algebra and by the existence of a bounded approximate identity for $A(H)$ is equivalent to (P_2) . \square

Let us summarize the results of this section as well as the previous one in the following theorem.

Theorem 3.11 *Let H be a regular Fourier hypergroup. Where*

(SF) H satisfies the strong Følner condition.

(L_D) H satisfies the D -Leptin condition for some $D \geq 1$.

(B_D) $A(H)$ has a D -bounded approximate identity for some $D \geq 1$.

(P_2) H satisfies (P_2) .

(P_1) H satisfies Reiter condition.

(AM) $L^1(H)$ is an amenable Banach algebra.

Then

$$\begin{array}{ccccccc}
 (SF) & \implies & (L_1) & \implies & (P_2) & \not\Rightarrow & (P_1) \not\Leftarrow (AM) \\
 & & \Downarrow & & \Uparrow & & \\
 & & (L_D) & \implies & (B_D) & \Longleftrightarrow & (B_1)
 \end{array}$$

Recall that the implication $(L) \Rightarrow (P_2)$ is due to [40, Proposition 4.4.3], as mentioned before.

Note. For a locally compact group H in Theorem 3.11, all aforementioned conditions are equivalent and equal the amenability of the group.

3.2 Bounded approximate identities in ideals of $A(H)$

For each set $E \subset H$, let us define $I_E := \{u \in A(H) : u(x) = 0 \ \forall x \in E\}$ which is an ideal of $A(H)$ if H is a regular Fourier hypergroup. Regarding bounded approximate identities of the ideals I_E , one may have the following observations. One may notice that some of these observations were proved in [9, 10] for commutative regular Fourier hypergroups.

- (i) Let H be a discrete regular Fourier hypergroup. Then every ideal which is finite dimensional has an identity. To observe this, for a finite dimensional ideal I , there is a finite set $F \subseteq H$ such that $I = \text{span}\{\delta_x\}_{x \in F}$. Hence, $u_F := \sum_{x \in F} \delta_x$ is the identity of I .
- (ii) For a discrete regular Fourier hypergroup satisfying (P_2) (or equivalently $A(H)$ has a bounded approximate identity), every ideal which is finite co-dimensional has a bounded approximate identity. To see this, assume that I is finite co-dimensional. Therefore, $I = I_F$ for some finite set $F \subseteq H$. suppose that $(u_\alpha)_\alpha$ is a bounded approximate identity of $A(H)$. For each α , define $v_\alpha := u_\alpha - \sum_{x \in F} u_\alpha(x) \delta_x \in I$. Hence, $(u_\alpha)_\alpha$ is an approximate identity of I while for each α ,

$$\|v_\alpha\|_{A(H)} \leq \|u_\alpha\|_{A(H)} + \sum_{x \in F} \|u_\alpha\|_\infty \|\delta_x\|_{A(H)}.$$

Note that for each $x \in H$, $\|\delta_x\|_{A(H)} \leq \|\delta_x\|_2 = h(x)^{1/2}$. So if $\sup_\alpha \|u_\alpha\|_{A(H)} = 1$,

$$\|v_\alpha\|_{A(H)} \leq 1 + \sum_{x \in F} h(x)^{\frac{1}{2}}.$$

- (iii) If H is a regular Fourier hypergroup (not necessarily discrete) and K is a compact open sub-hypergroup of H then I_K has the identity 1_K . It is a direct result of Lemma 1.3 where $1_K = 1_K *_h 1_K \in A(H)$ which is constantly 1 on K .
- (iv) Let H be a regular Fourier hypergroup satisfying (P_2) and $(u_\alpha)_\alpha$ is a 1-bounded approximate identity of $A(H)$. Let us define $\mathcal{G}(H) := \{x \in H : x * \tilde{x} = e\}$ which is the **maximal group** of the hypergroup H . Then $I_{\{x\}}$, for each $x \in \mathcal{G}(H)$, has a bounded approximate identity. To generate a bounded approximate identity of $I_{\{x\}}$, for each V which belongs to \mathcal{N}_e , the neighbourhoods of e directed inversely by inclusion, define $\varepsilon_V = 1_{xV} *_h \tilde{1}_V \in A(H)$ (see Lemma 1.3). So $\varepsilon_V(x) = 1$ and for each $y \neq x$ there is some $V_0 \in \mathcal{N}_e$ such that for every $V \subseteq V_0$, y does not belong to the closure of $x * V * \tilde{V} = \text{supp } \varepsilon_V$ while $\|\varepsilon_V\|_{A(H)} \leq 1$ for each V . So for each $V \in \mathcal{N}_e$ and α define $e_{\alpha,V} := u_\alpha - u_\alpha(x) \varepsilon_V \in I_{\{x\}}$. Now one easily sees that the net $(e_{\alpha,V})_{\alpha,V \in \mathcal{N}_e}$ is a 2-bounded approximate identity of $I_{\{x\}}$.

Remark 3.12 Note that in the observation (iv), one cannot immediately prove something similar for every $x \in H$, since for $\varepsilon_{x,V} = 1_{xV} *_h \tilde{1}_V$,

$$\|\varepsilon_{x,V}\|_{A(H)} \leq \left(\frac{\lambda_H(x * V)}{\lambda_H(V)} \right)^{\frac{1}{2}}.$$

But for a general $x \in H$, $\lambda_H(x * V) \geq \lambda_H(V)$ while for $x \in \mathcal{G}(H)$ one gets the equality. Therefore, although the reasoning of the observation (iv) implies the existence of an approximate identity for every I_x , it does not guarantee that the approximate identity is bounded unless $x \in \mathcal{G}(H)$.

Question. Let H be a regular Fourier hypergroup such that $A(H)$ has a bounded approximate identity. For which subsets $E \subseteq H$, does I_E have a bounded approximate identity?

A complex valued function ϕ on H is called a **multiplier** of $A(H)$, if ϕu lies in $A(H)$ whenever u belongs to $A(H)$. Let us denote by $MA(H)$ the space of all multipliers of $A(H)$ as defined and studied in [34]. Then [34, Proposition 3.2] proves that every $\phi \in MA(H)$ is continuous. Also clearly $MA(H)$ contains constant functions. Even more, $MA(H)$ is a Banach algebra on functions on H equipped with the operator norm of $\mathcal{B}(A(H), A(H))$. The following proposition is a hypergroup version of [14, Lemma 3.9].

Proposition 3.13 *Let H be a regular Fourier hypergroup such that $A(H)$ satisfying (P_2) . If A and B are two disjoint subsets of H such that there is some $u \in MA(H)$ such that $u|_A \equiv 1$ and $u|_B \equiv 0$. Then $I_{A \cup B}$ has a bounded approximate identity if and only if $I(A)$ and $I(B)$ have bounded approximate identities.*

Proof. Let $(u_\alpha)_\alpha$ be a bounded approximate identity of $I_{A \cup B}$ and (v_β) be a bounded approximate identity of $A(H)$ proved by Theorem 3.7. Then $(u_\alpha u + (1 - u)v_\beta)_{\alpha,\beta}$ is a bounded approximate identity of I_A . Similarly $(u_\alpha(1 - u) + uv_\beta)_{\alpha,\beta}$ is a bounded approximate identity of I_B . The converse is proved by multiplying the bounded approximate identities of I_A and I_B and it is independent of the existence a bounded approximate identity of $A(H)$ as well as the existence of such a u in $MA(H)$. \square

4 Amenability of hypergroup algebras

In this section H is always a discrete hypergroup with a fixed Haar measure h . Let us first recall that $L^1(H, h)$ is a subalgebra of $\ell^1(H)$. Also, they are isometrically Banach algebra isomorphic through a mapping $\iota : L^1(H, h) \rightarrow \ell^1(H)$ where $\delta_x \mapsto h(x)\delta_x$. Therefore, for $\mathcal{F} : L^1(H, h) \rightarrow C(\hat{H})$ and $\mathcal{FS} : \ell^1(H) \rightarrow C(\hat{H})$, the Fourier and Fourier-Stieltjes transforms, respectively (see Section 1), $\mathcal{F}(\delta_x) = h(x)\mathcal{FS}(\delta_x)$ for each $x \in H$. Hence $\mathcal{FS}(\iota(f)) = \mathcal{F}(f)$ for

every $f \in L^1(H, h)$. Since the same holds for the hypergroup $H \times H$, one may consider the corresponding Fourier and Fourier-Stieltjes transforms, denoted by \mathcal{F}_2 and \mathcal{FS}_2 respectively. Let us recall that, $L^1(H \times H, h \times h)$ is isometrically Banach algebra isomorphic to $L^1(H, h) \otimes_\gamma L^1(H, h)$ where \otimes_γ denotes the projective tensor product. Therefore, $\mathcal{F}_2(f \otimes g)(\alpha, \beta) = \mathcal{F}(f)(\alpha)\mathcal{F}(g)(\beta)$ for every $\alpha, \beta \in \widehat{H}$. A similar result holds for \mathcal{FS}_2 . To prove the main theorem of this section we need the following lemma.

Lemma 4.1 *Let H be a discrete commutative hypergroup. If $L^1(H, h)$ is amenable then there is a constant $\epsilon > 0$ such that for every $\alpha, \beta \in \widehat{H}$, there is a function $g \in c_c(H)$ such that $|\mathcal{F}(g)(\alpha) - \mathcal{F}(g)(\beta)| > \epsilon$.*

Proof. If $L^1(H, h)$ is amenable, by [15], one can separate the elements of the Gelfand spectrum by the elements of $L^1(H, h)$. Therefore, there is a constant $\varepsilon > 0$ such that for every $\alpha, \beta \in \widehat{H}$ such that $\alpha \neq \beta$, there exists some $f (= f_{\alpha, \beta}) \in L^1(H, h)$ such that $|\mathcal{F}(f)(\alpha) - \mathcal{F}(f)(\beta)| > \varepsilon$. Since $c_c(H)$ is dense in $L^1(H, h)$, there is some $g \in c_c(H)$ such that $\|g - f\|_1 < \varepsilon/4$. Therefore,

$$\begin{aligned} \frac{\varepsilon}{2} &\geq |\mathcal{F}(f)(\alpha) - \mathcal{F}(g)(\alpha) - \mathcal{F}(f)(\beta) + \mathcal{F}(g)(\beta)| \\ &\geq |\mathcal{F}(f)(\alpha) - \mathcal{F}(f)(\beta)| - |\mathcal{F}(g)(\alpha) - \mathcal{F}(g)(\beta)| \\ &\geq \varepsilon - |\mathcal{F}(g)(\alpha) - \mathcal{F}(g)(\beta)|. \end{aligned}$$

And this proves the lemma for $\epsilon = \varepsilon/2$. □

The following theorem is the main theorem of this section. The whole idea behind the proof is from a study on weighted group algebras, [5]. Due to the similarity of the Haar measure of discrete hypergroups and weights on discrete groups, Lasser in [30, Theorem 3] applied a similar argument to prove the following result for polynomial hypergroups. Here we prove Lasser's result for discrete commutative hypergroups satisfying (P_2) . Let us recall that for a commutative hypergroup H with the Plancherel measure ϖ , H satisfies (P_2) if and only if the constant character 1 belongs to $\text{supp}(\varpi)$, [41].

Theorem 4.2 *Let H be an infinite discrete commutative hypergroup which satisfies (P_2) . If $L^1(H)$ is amenable then there is some M such that $\{x : h(x) \leq M\}$ is infinite.*

Proof. Let us denote by $c_0(H \times H)$ the Banach subspace of $\ell^\infty(H \times H)$ vanishing at infinity. Indeed $c_0(H, H)$ forms an $L^1(H, h)$ -bimodule applying the following definitions $f \cdot \phi := f \otimes \delta_e * \phi$ and $\phi \cdot f := f \otimes \delta_e * \phi$ for every $f \in L^1(H, h)$ and $\phi \in c_0(H \times H)$ where e is the trivial element of H . Proposition 1.2 implies that the outcome also belongs to $c_0(H \times H)$. By Riesz representation theorem the dual of $c_0(H \times H)$ is nothing but $\ell^1(H \times H)$. But recall that $L^1(H \times H, h \times h)$ is isomorphic to $\ell^1(H \times H)$ through the mapping $\delta_{(x, y)} \mapsto h(x)h(y)\delta_{(x, y)}$. Therefore, for every

$\phi \in c_0(H \times H)$ and $\Phi \in L^1(H \times H, h \times h)$,

$$\langle \Phi, \phi \rangle = \sum_{x, y \in H} \phi(x, y) \Phi(x, y) h(x) h(y).$$

Now one may consider $L^1(H \times H, h \times h)$ as a dual Banach $L^1(H, h)$ -bimodule.

$$\begin{aligned} \langle \phi, \Phi \cdot f \rangle &= \langle f \cdot \phi, \Phi \rangle \\ &= \sum_{x, y \in H} \sum_{z \in H} f(z) \phi(\delta_z * \delta_x, y) h(z) \Phi(x, y) h(x) h(y) \\ &= \sum_{x, y \in H} \sum_{z \in H} f(z) \Phi(\delta_z * \delta_x, y) h(z) \phi(x, y) h(x) h(y) \\ &= \sum_{x, y \in H} \sum_{z \in H} \tilde{f}(z) \Phi(\delta_z * \delta_x, y) h(z) \phi(x, y) h(x) h(y) \\ &= \langle \phi, \tilde{f} \otimes \delta_e * \Phi \rangle. \end{aligned}$$

Similarly, $f \cdot \Phi = \delta_e \otimes \tilde{f} * \Phi$. Toward a contradiction assume that $h(x) \rightarrow \infty$ and $L^1(H, h)$ is amenable. Let $\phi_0 \in c_0(H \times H)$ be defined as $\phi_0(x, y) = h(x)^{-1} h(y)^{-1}$. Therefore, $\mathcal{X} := \text{Ker}(\phi_0)$ is the weak* closed subspace of $\ell^1(H \times H)$ consisting of all Φ such that $\sum_{x, y} \Phi(x, y) = 0$. In particular for each $\Phi \in L^1(H \times H, h \times h) \subseteq \ell^1(H \times H)$, $\mathcal{FS}(\Phi)(1, 1) = \sum_{x, y} \Phi(x, y) = 0$ if $\Phi \in \text{Ker}(\phi_0)$. Also, for each $f \in L^1(H, h)$, $\mathcal{FS}(f \cdot \Phi)(1, 1) = \mathcal{FS}(f)(1) \mathcal{FS}(\Phi)(1, 1) = 0$ and similarly, $\mathcal{FS}(\Phi \cdot f)(1, 1) = 0$. Hence, since \mathcal{X} is weak* closed, \mathcal{X} forms a dual $L^1(H, h)$ -bimodule, see [5, Proposition 1.3]. Let us define a mapping $D : L^1(H, h) \rightarrow \mathcal{X}$ by $Df := f \otimes \delta_e - \delta_e \otimes f$. It is clear that $\mathcal{FS}(Df)(1, 1) = 0$ and $D(f) = f \cdot \delta_{e, e} - \delta_{e, e} \cdot f$ for every $f \in L^1(H, h)$; D is a derivation. (Note that based on the formula of D with respect to $\delta_{(e, e)}$ before, D is not necessarily an inner derivation into \mathcal{X} .)

If $L^1(H, h)$ is amenable, then there is some $\Phi \in \mathcal{X}$ such that $Df = f \cdot \Phi - \Phi \cdot f$ for every f . Also by Lemma 4.1, there exists some $\epsilon > 0$ such that for every $\alpha, \beta \in \widehat{H}$ when $\alpha \neq \beta$ there is some $f (= f_{(\alpha, \beta)}) \in c_c(H)$ such that $|\mathcal{F}(f)(\alpha) - \mathcal{F}(f)(\beta)| > \epsilon$. Therefore, for $g = \iota(f) \in c_c(H)$ we have

$$\begin{aligned} 0 \neq \mathcal{F}(f)(\alpha) - \mathcal{F}(f)(\beta) &= \mathcal{FS}(g)(\alpha) - \mathcal{FS}(g)(\beta) \\ &= \mathcal{FS}_2(Dg)(\alpha, \beta) \\ &= \mathcal{FS}(g)(\alpha) \mathcal{FS}(\Phi)(\alpha, \beta) - \mathcal{FS}(g)(\beta) \mathcal{FS}_2(\Phi)(\alpha, \beta) \\ &= (\mathcal{FS}(g)(\alpha) - \mathcal{FS}(g)(\beta)) \mathcal{FS}_2(\Phi)(\alpha, \beta) \\ &= (\mathcal{F}(f)(\alpha) - \mathcal{F}(f)(\beta)) \mathcal{FS}_2(\Phi)(\alpha, \beta). \end{aligned}$$

Therefore, $\mathcal{FS}_2(\Phi)(\alpha, \beta) = 1$ for all $\alpha \neq \beta$ while $\mathcal{FS}_2(\Phi)(1, 1) = 0$. The continuity of $\mathcal{FS}_2(\Phi)$ implies that $(1, 1) \in \widehat{H} \times \widehat{H}$ is an isolated point. Since H and consequently $H \times H$ satisfy (P_2) ,

$(1, 1) \in \text{supp}(\varpi)$; therefore, $\varpi(1, 1) > 0$ where ϖ denotes the Plancherel measure on $\widehat{H} \times \widehat{H}$. So, $0 \neq \delta_{(1,1)} \in L^2(\widehat{H} \times \widehat{H}, \varpi)$. But by Plancherel theorem for commutative hypergroups (see [7, Theorem 2.2.2]), $\mathcal{F}_2^{-1}(\delta_{(1,1)}) \neq 0$ belongs to $L^2(H, h)$. Also based on the definition of the Fourier inverse, [7, Definition 2.2.30],

$$\Psi(x, y) := \mathcal{F}_2^{-1}(\delta_{(1,1)})(x, y) = \sum_{\alpha, \beta \in \widehat{H}} \delta_{(1,1)}(\alpha, \beta) \overline{\alpha(x)} \overline{\beta(y)} \varpi(\alpha, \beta) = \varpi(1, 1).$$

Therefore, the constant function $\Psi \equiv \varpi(1, 1)$ belongs to $L^2(H \times H, h \times h)$ which contradicts our assumption regarding the unboundedness of h . \square

In the remaining of this section, we prove the amenability of the hypergroup algebra of **multivariable Chebychev polynomial hypergroups of the first kind**. This hypergroup structure is defined on \mathbb{N}_0^d for some integer $d \geq 1$. Let us recall that the convolution action for any (n_1, \dots, n_d) and (m_1, \dots, m_d) in \mathbb{N}_0^d is defined

$$\delta_{(n_1, \dots, n_d)} * \delta_{(m_1, \dots, m_d)} = \frac{1}{2^d} \sum \delta_{(|\pm n_1 \pm m_1|, \dots, |\pm n_d \pm m_d|)} \quad (4.1)$$

when the sigma is taken over all 2^d possibilities of $\pm n_i$ and $\pm m_j$ for $1 \leq i, j \leq d$.

For $d = 1$, this hypergroup is simply called **Chebychev polynomial hypergroup of first kind** and the amenability of the hypergroup algebra is proved in [30]. In his proof, Lasser constructs a bounded approximate diagonal of the hypergroup algebra. The following proof though, applies some results for amenable algebras and one observation on hypergroup algebra of this class of polynomial hypergroups to generalize Lasser's result. The author appreciate Yemon Choi and Nico Spronk's help for the following proposition.

Proposition 4.3 *The hypergroup algebra of the multivariable Chebychev polynomial hypergroups is amenable.*

Proof. Let \mathbb{T} denote the torus group and $\mathbb{F}_2 = \{e, \alpha\}$ be the group of order two of automorphisms of \mathbb{T} where $\alpha(\theta) = \theta^{-1}$. Hence, \mathbb{F}_2^d is a subgroup of automorphisms of \mathbb{T}^d for any integer $d \geq 1$. Let us recall that the dual group of \mathbb{T}^d is nothing but \mathbb{Z}^d and $A(\mathbb{T}^d)$, the Fourier algebra of \mathbb{T}^d , is isometrically Banach algebra isomorphism to $\ell^1(\mathbb{Z}^d)$ and therefore is amenable. Let $\chi_{(n_1, \dots, n_d)}$ be the Fourier transform of $\delta_{(n_1, \dots, n_d)} \in \ell^1(\mathbb{Z}^d)$.

One may consider $Z_{\mathbb{F}_2^d} A(\mathbb{T}^d)$ which is the subalgebra of $A(\mathbb{T}^d)$ consisting of all functions which are invariant with respect to the group \mathbb{F}_2^d which forms a finite group of automorphisms for the algebra $A(\mathbb{T}^d)$. One simple observation implies that this algebra is generated by all characters of the form $\psi_{(n_1, n_2, \dots, n_d)} := \sum \chi_{(\pm n_1, \dots, \pm n_d)}$ for all $(n_1, \dots, n_d) \in \mathbb{N}_0^d$ when the sigma is taken over all 2^d possibilities of $\pm n_i$ for $1 \leq i \leq d$. Since $\|\psi_{(n_1, \dots, n_d)}\|_{A(\mathbb{T}^d)} = d$ for every (n_1, \dots, n_d) and

regarding the convolution (4.1), one can easily show that $Z_{\mathbb{F}_s^d}A(\mathbb{T}^d)$ is isometrically isomorphic to the d -variable Chebychev polynomial hypergroup algebra.

Note that as an abelian group, the Fourier algebra of \mathbb{T}^d is amenable. Also via a result by Kepert, in [26], for every finite group of automorphisms of an amenable algebra, the subalgebra of all invariant elements is also amenable. And $Z_{\mathbb{F}_2^d}A(\mathbb{T}^d)$ is such a subalgebra for amenable algebra $A(\mathbb{T}^d)$; hence, it is amenable. \square

Remark 4.4 Note that for each d -variable Chebychev polynomial hypergroup \mathbb{N}_0^d , the hypergroup algebra satisfies (P_2) , [28], and the Haar measure is constantly 2^d . Comparing this observation and Theorem 4.2, one may conjecture that for this family of hypergroups, the amenability should be equivalent to the boundedness of the Haar measure. In a subsequent work, we study this conjecture for dual of hypergroups.

5 Applications to compact and discrete groups

5.1 Approximate amenability of proper Segal algebras

In [1], it was shown that for every proper Segal algebra of a compact group G is not approximately amenable if \widehat{G} satisfies the Leptin condition. The proof of [1, Theorem 5.3] is also correct for all proper Segal algebras on compact group G when \widehat{G} satisfies the D -Leptin condition for some $D \geq 1$. Basically, the D -Leptin condition helps us to generate a norm bounded approximate identity for the Fourier algebra of \widehat{G} which satisfies some extra conditions. Here we omit the proof as it is identical to the one in [1].

Theorem 5.1 *Let G be a compact group and \widehat{G} satisfies the D -Leptin condition for some $D \geq 1$. Then every proper Segal algebra of G is not approximately amenable.*

Corollary 5.2 *Every proper Segal algebra on every connected simply connected compact real Lie group is not approximately amenable.*

5.2 Amenability of $ZA(G)$ for compact groups

It was known that every compact group is a regular Fourier hypergroup, [18]. Let us denote by $ZA(G)$, the subspace of $A(G)$, of all functions f such that f is constant on every conjugacy class. Therefore,

$$ZA(G) = \{f \in A(G) : f(yxy^{-1}) = f(x) \forall x, y \in G\} \quad (5.1)$$

which forms a closed subspace of $A(G)$.

Theorem 5.3 *Let G be a non-discrete compact group such that $\{\pi \in \widehat{G} : d_\pi = n\}$ is finite for each positive integer n . Then $ZA(G)$ is not amenable.*

Proof. In [1], it was proved that $ZA(G)$, as a Banach algebra, is isometrically isomorphic to the hypergroup algebra of \widehat{G} where $h(\pi) = d_\pi^2$ for every $\pi \in \widehat{G}$. Moreover, \widehat{G} is a regular Fourier algebra and its Fourier algebra isometrically isomorphic to $ZL^1(G)$, the center of the group algebra of G . But as a SIN group, $ZL^1(G)$ has a 1-bounded approximate identity. Hence, $A(\widehat{G})$ has a bounded approximate identity; therefore, by Theorem 3.7, \widehat{G} satisfies (P_2) . Now one can apply Theorem 4.2 for the hypergroup \widehat{G} to finish the proof. \square

One may compare Theorem 5.3 to [24, Theorem 6.1] where Johnson proved a similar result for amenability of $A(G)$. In his proof, though, he used a property of a subalgebra of $A(G)$ denoted by $A_\gamma(G)$. The author's computations imply that such a procedure fails for hypergroups. One may see [42] for a survey on the amenability notions of $A(G)$.

Remark 5.4 The condition of Theorem 5.3 is far from being necessary for the amenability of $ZA(G)$ as Johnson, in [24], stated a similar remark for $A(G)$. For example let $G = \mathbb{T} \times \text{SU}(2)$. One can show that $ZA(\text{SU}(2))$ is not even weakly amenable because it has a non-zero bounded inner derivation, say D_θ . Therefore, $D_\theta \otimes \varepsilon_e$ forms a symmetric non-zero bounded derivation on $ZA(\text{SU}(2)) \otimes_\gamma A(\mathbb{T})$ when $\varepsilon_e(g) := g(e)$ for e the identity of the group \mathbb{T} ; hence, $ZA(\text{SU}(2) \times \mathbb{T}) \cong ZA(\text{SU}(2)) \otimes_\gamma A(\mathbb{T})$ is not weakly amenable. On the other hand, for each n there are infinitely many $\pi \in \widehat{G}$ such that $d_\pi = n$. The details of this remark will appear in a subsequent paper which is currently in preparation.

Remark 5.5 Let G be a compact group such that $d_\pi \rightarrow \infty$. Then G is called a **tall group**. Some properties of tall groups, specially profinite tall groups, have been studied in [21, 22, 37].

Example 5.6 Let us use the notations and facts mentioned in the proof of Proposition 2.12. So based on Weyl's formula and applying a rough estimate, one can easily show that $\tau(\pi) \leq |B|d_\pi$. Hence, $\{d_\pi\}_{\pi \in \widehat{G}}$ cannot be bounded when B is finite. Therefore, $ZA(\mathbb{G})$ is not amenable. This class of compact Lie groups includes $\text{SU}(n)$'s for $n \geq 2$.

5.3 Amenability of $Z\ell^1(G)$ for FC groups

In this subsection, we are interested in applying the hypergroup tools that we have developed before to study the amenability of $Z\ell^1(G)$ for discrete FC groups. In this section for locally compact group G , $\text{Aut}(G)$ is the group of all bicontinuous automorphisms of G . Recall that G is a $[FIA]^{-B}$ group provided B is a subgroup of $\text{Aut}(G)$ which has compact closure in $\text{Aut}(G)$ with respect to the **Birkhoff topology**. Let us assume that B always includes the inner automorphisms. With a natural operation the set $G_B^\#$ of B -orbits $C_x^B := \{\beta(x) : \beta \in B\}$, $x \in G$, is a commutative hypergroup, see [23, 8.1]. When B collapses to the subgroup of all inner automorphisms, $[FIA]^{-B}$ is denoted by $[FIA]$ and $G_B^\#$ simply by $G^\#$. Here, we rely on some results of [33]

regarding $[FIA]^\bar{B}$ groups. Recall that for a locally compact group G , B as a subgroup of $\text{Aut}(G)$ has a compact closure in $\text{Aut}(G)$ if and only if G is a $[SIN]$ group and G is $[FC]$ (that is, the conjugacy classes of G have compact closure). The two concepts $[FC]$ and $[FIA]$ are equivalent for discrete groups.

Let $Z_B L^p(G)$ be as defined by Mosak in [33] for $p \in [1, \infty)$. It can be shown that the canonical natural map $\iota : Z_B L^p(G) \rightarrow L^p(G_B^\#)$ where $\iota(f)(C_x^B) := f(x)$ is an isometric Banach space isomorphism. Moreover, if $p = 1$, the mapping ι is also a Banach algebra isomorphism. On the other hand, $G_B^\#$ forms a regular Fourier hypergroup. One may see [38, 34] for more details regarding these claims. Specially to prove that $A(G_B^\#)$ is an algebra, one may apply [17] which proves that the dual structure of $G_B^\#$ is another hypergroup denoted by H_B here. Therefore, $A(G_B^\#)$ is isometrically Banach algebra isomorphism to $L^1(H_B)$ and hence $A(G_B^\#)$ has a bounded approximate identity, by Proposition 3.3.

Remark 5.7 Let $\iota^* : C_c(G_B^\#) * C_c(G_B^\#) \rightarrow Z_B C_c(G) * Z_B C_c(G)$ be the canonical restriction of ι^{-1} to compact supported functions where $Z_B C_c(G)$ is $C_c(G) \cap Z_B L^1(G)$. Therefore, for each $u \in C_c(G_B^\#) * C_c(G_B^\#) \subseteq A(G_B^\#)$,

$$\begin{aligned} \|u\|_{A(G_B^\#)} &= \inf\{\|\xi\|_2 \|\eta\|_2 : u = \xi * \tilde{\eta}, \quad \xi, \eta \in L^2(G_B^\#)\} \\ &\geq \inf\{\|\xi\|_2 \|\eta\|_2 : \iota^*(u) = \xi * \tilde{\eta}, \quad \xi, \eta \in L^2(G)\} = \|\iota^*(u)\|_{A(G)}. \end{aligned}$$

Therefore, ι^* can be extended to a norm decreasing linear mapping $\iota^* : A(G_B^\#) \rightarrow Z_B A(G)$ for $Z_B A(G) := \{f \in A(G) : f \circ \beta = f, \forall \beta \in B\}$. Noting this fact that both of the algebras $A(G_B^\#)$ and $Z_B A(G)$ are equipped with pointwise multiplication, one can conclude that ι^* is an algebra homomorphism. Due to the definition of ι^* over compact supported functions of $G_B^\#$ into $Z_B C_c(G)$, ι^* is an injection with a dense range, even though it is not immediate that ι^* is a bijection necessarily. Although this is true for every compact group G and B the group of inner automorphisms, [1]. It is an interesting question that for which $[FIA]^\bar{B}$ groups these two algebras are isomorphic.

Let us recall again that for a discrete group G , it is called **FC** or **finite conjugacy** if for each $x \in G$, $C_x := \{yxy^{-1} : y \in G\}$ is finite. Every FC group G is actually $[FIA]$. Note that for an FC group G , $G^\#$ is a discrete commutative hypergroup. Moreover, then the weight h which is defined on $G^\#$ by $h(C) = |C|$, ($C \in G^\#$), is a Haar measure on $G^\#$, [2].

Theorem 5.8 *Let G be an infinite FC group such that for every integer n there are just finitely many conjugacy classes C such that $|C| = n$. Then $Z\ell^1(G)$ is not amenable.*

Proof. As we saw before, $G^\#$ is a regular Fourier discrete commutative hypergroup. Also, $A(G^\#)$ has a bounded approximate identity and therefore, $G^\#$ satisfies (P_2) by Theorem 3.7. Now one

applies Theorem 4.2 and isomorphism $L^1(G^\#, h) \cong Z\ell^1(G)$ to finish the proof. \square

For a group G , let G' denote the **derived subgroup** of G . It is immediate that if G' is finite, for every $C \in G^\#$, $|C| \leq |G'|$. The converse is also true i.e. if $\sup_{C \in G^\#} |C| < \infty$, then $|G'| < \infty$.

Remark 5.9 Recall that in [4], it is proven that for a locally compact group G with a finite derived subgroup G' , $Z\ell^1(G)$ is amenable. Also for a specific class of FC groups, called RDPF groups, the amenability of $Z\ell^1(G)$ is characterized in [3]. The result is that, for a RDPF group G , $Z\ell^1(G)$ is amenable if and only if $|G'| < \infty$. These two studies suggest that the later characterization for RDPF groups may be extendible to general FC groups.

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Mahmood Alaghmandan

The Fields Institute For Research In Mathematical Sciences,
222 College St,
Toronto, ON M5T 3J1, Canada

&

Department of Pure Mathematics,
University of Waterloo,
Waterloo, ON N2L 3G1, Canada

Email: m.alaghmandan@utoronto.ca